TÍTULO: Problema directo e inverso del tipo de Cauchy para la ecuación de fracciones con la degeneración y el coeficiente del operador sin carga.

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RESUMEN: Se demuestra una condición suficiente para la solvencia de un problema de tipo Cauchy para una ecuación abstracta degenerada de orden fraccional. Una característica distintiva del trabajo es el hecho de que el dominio de la determinación del coeficiente del operador, que caracteriza la degeneración, no es tan denso. De acuerdo con los coeficientes de operación de la ecuación, desarrollan una función de operador que pertenece al espacio de los operadores lineales delimitados, y con su ayuda se establece una solvencia única de problemas directos e inversos de tipo Cauchy. Notemos que el problema inverso degenerado se considera por primera vez.

PALABRAS CLAVES: ecuación fraccionaria degenerada, operador no definido densamente, problema de tipo Cauchy directo e inverso, operador de resolución, resolución inequívoca.

TITLE: Direct and inverse problem of Cauchy type for the fraction equation with the degeneration and non-loaded operator coefficient.

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ABSTRACT: A sufficient condition is proved for the solvability of a Cauchy-type problem for a degenerate fractional-order abstract equation. A distinctive feature of the work is the fact that the domain of the operator coefficient determination, which characterizes the degeneracy, is not assumed to be dense one. According to the operator coefficients of the equation they develop some operator function belonging to the space of linear bounded operators, and with its help a unique solvability of direct and inverse problems of Cauchy type is established. Let's note that the degenerate inverse problem is considered for the first time.

KEY WORDS: degenerate fractional equation, non-densely defined operator, direct and inverse Cauchy type problem, resolving operator, unambiguous resolvability.

INTRODUCTION.

In the Banach space E, we consider a linear differential equation of fractional order with a degenerate operator at a fractional derivative. By the operator coefficients of the equation, some operator-valued function $U(t)$ is developed that belongs to the space $L(E)$ of bounded linear operators acting from E into E, and it helps to establish a unique solvability of the direct and the inverse problems of Cauchy type.

Let A and B be the linear operators with the domains of definition $D(A)$ and $D(B)$ respectively, at that $D(B) \subset D(A)$. Let's introduce A-resolvent set $\rho_A(B) = \left\{ \lambda \in C : (\lambda A - B)^{-1} \in L(E) \right\}$ for consideration. On this set the operator function $(\lambda A - B)^{-1}$ is determined – the generalized resolvent of the operator B.

Term 1. We shall assume that the half-plane $\operatorname{Re} \lambda \geq 0$ of the complex plane C is contained in the A-resolvent set, that is $\{\lambda \in C : \operatorname{Re} \lambda \geq 0\} \subset \rho_A(B)$. Moreover, suppose that at $\lambda = \sigma + i\tau$ the generalized resolvent $(\lambda A - B)^{-1}$ of the operator B at $\lambda \to \infty$ is defined in a region wider than the half-plane of the area Ξ, bounded to the left by the curve γ with the equation $\sigma = \omega - \omega_0 (1 + \tau^2)^{r/2}$, $\tau \in (-\infty, \infty)$, $\omega > \omega_0 > 0$ and in this domain with some constant $M > 0$ the following estimation is performed:

$$\left\| A(\lambda A - B)^{-1} \right\| \leq \frac{M}{|\lambda|^r}, \quad \lambda \in \Xi, \quad \frac{2}{3} < r < 1. \tag{1}$$

Term 1 implies, in particular, the existence of a bounded inverse operator B^{-1}. The product of the operator A on a generalized resolvent does not have a maximal order of the resolvent decrease equal to 1, but the estimate (1) allows us to develop the operator function $U(t)$ indicated above. Let's note also that the operator B is not necessarily defined densely, and this fact implies a special form of the initial condition for the equation considered below.

Let's consider a Cauchy-type problem at $0 < \alpha < 1$

$$AD^{\alpha} u(t) = Bu(t), \quad t > 0, \tag{2}$$

$$\lim_{t \to 0} AI^{1-\alpha} u(t) = AB^{-1} u_0, \tag{3}$$

where $\ker A \neq \{0\}$, which means the degeneration of the equation (2), $D^\alpha u(t) = \left(\dfrac{d}{dt}\right)I^{1-\alpha}u(t)$ – the

left-hand fractional Riemann-Liouville derivative, and $I^{1-\alpha}u(t) = \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \dfrac{u(\tau)}{(t-\tau)^\alpha}d\tau$ is the left-

sided fractional Riemann-Liouville integral, $\Gamma(\cdot)$ is the gamma function.

Definition 1. The solution of the problem (2), (3) is the function $u(t) \in C((0,\infty), D(B))$, for which

$I^{1-\alpha}u(t) \in C^1((0,\infty), D(A))$, $AD^\alpha u(t)$, $Bu(t) \in C((0,\infty), E)$, and satisfying the equation (2) and the

initial condition (3).

Among the papers devoted to the study of fractional-order equations with the degeneracy under various assumptions, let's mention the papers (Balachandran, Kiruthika, 2012; Li, Liang, Xu, 2012; Fedorov, Debbouche, 2013; Fedorov, et al. 2015). For example, in (Balachandran, Kiruthika, 2012), (Li, Liang, Xu, 2012), the operator A is imposed with the condition of continuous invertibility, and in (Fedorov, Debbouche, 2013; Fedorov, et al. 2015) the case of a degenerate operator A is considered and, according to the terminology of the authors, of a strongly-sectorial (L, p) or bounded (L, p) operator B.

In contrast to the listed works, during the development of problem (2), (3) solution, the domain of the operator A definition is not assumed to be dense, and other conditions, except for the estimate (1) to the operators A and B, as well as the Banach space E, are not superimposed.

The first-order equations in the case of a non-dense domain for the determination of operator coefficients were considered earlier in (Da Prato, Sinestrari, 1987; Favini, Yagi, 1995). Under the assumptions made for the operators A and B, the direct Cauchy problem for a degenerate first-order equation was studied in (Sil'chenko, 2002). In particular, an estimate of the form (1) is valid for a series of differential operators with non-=regular boundary conditions see (Sil'chenko, 1997) In (Sil'chenko, 2002), as an example of the operators A and B, satisfying the term 1, the following

operators are given in the Banach space $E = L_p(0,1) \times L_p(0,1)$: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\qquad B = \begin{pmatrix} \dfrac{d^2}{dx^2} & -\dfrac{d}{dx} \\ -\dfrac{d}{dx} & -1 \end{pmatrix}$

with the determination domains $D(A) = E$,

$$D(B) = \left\{ (u(x), v(x))^T \; : \; u(x) \in W_p^2(0,1), \; v(x) \in W_p^1(0,1), \; u(0) = 0, \; \int_0^1 u(x)\,dx = 0 \right\}.$$

As was established in (Sil'chenko, 2002), the estimation (1) was performed for the operators A and

B at $r = \dfrac{1}{2} + \dfrac{1}{2p}$, where $1 < p < 3$.

DEVELOPMENT.

Study methods.

The study of abstract equations with fractional derivatives is performed by the methods of operator

theory and operator special functions. Various operator relations, a number of operator characteris-

tics (spectrum, resolvent), the basic theorems of functional analysis, etc. are used. A big role is as-

signed to special functions.

Main results.

1. Cauchy type problem. Let's introduce the operator function.

$$U(t) = \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) A(\lambda A - B)^{-1}\,d\lambda, \tag{4}$$

which contains the function of Mittag-Leffler type under the integral sign

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0, \quad \beta > 0.$$

Let's show that the integral in (4) is absolutely convergent. To do this, let's represent the Mittag-

Leffler function in the form (see formulas (2.2.31), (2.1.1) from (Pshu, 2005).

$$E_{\alpha,\alpha}(t^{\alpha}\lambda) = t^{-\alpha}\int_{0}^{\infty} e_{1,\alpha}^{1,0}(-st^{-\alpha})\, e^{\lambda s}\, ds, \tag{5}$$

where the Wright-type function (see (Pshu, A.V., 2005), Chapter 1) is defined by

$$e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k + \mu)\Gamma(\delta - \beta k)}, \quad \alpha > \max\{0;\beta\}, \quad \mu, z \in C.$$

For a Wright-type function $e_{1,\alpha}^{1,0}(-z)$ the following inequality holds (see (Pshu, 2005), Lemma 1.2.7)

$$e_{1,\alpha}^{1,0}(-z) \le C_{n}(z)\exp\!\left(-\rho z^{1/(1-\alpha)}\right), \quad z > 0, \quad \rho > 0, \tag{6}$$

where

$$C_{n}(z) = \sum_{k=0}^{n}\frac{\alpha^{k}z^{k}}{\Gamma(k - k\alpha)}, \qquad n \in N \cup\{0\}, \quad n \ge \frac{1}{1-\alpha}.$$

Taking (1), (4), (5) and the form of the integration contour γ at $\lambda = \sigma + i\tau$, $\sigma = \omega - \omega_{0}(1+\tau^{2})^{r/2}$

we have the following:

$$\|U(t)\| \le \frac{M_{0}}{t}\int_{-\infty}^{\infty}\int_{0}^{\infty} e_{1,\alpha}^{1,0}(-st^{-\alpha})\exp\!\left(\omega s - \omega_{0}s(1+\tau^{2})^{r/2}\right)\!\left(\sigma^{2}+\tau^{2}\right)^{-r/2}\left|\frac{d\sigma}{d\tau}+i\right|ds\,d\tau \le$$

$$\le \frac{C_{0}}{t}\int_{0}^{\infty}\int_{0}^{\infty} e_{1,\alpha}^{1,0}(-st^{-\alpha})\exp\!\left(\omega s - \omega_{0}s(1+\tau^{2})^{r/2}\right)\!\left(\sigma^{2}+\tau^{2}\right)^{-r/2}ds\,d\tau \le$$

$$\le \frac{M_{1}}{t}\int_{0}^{\infty}\int_{0}^{\infty} e_{1,\alpha}^{1,0}(-st^{-\alpha})\exp\!\left(\omega s - \omega_{0}s\tau^{r}\right)\frac{ds\,d\tau}{\tau^{r}}. \tag{7}$$

The estimate (6) ensures the absolute convergence of the integral in (7). Changing the order of integration, we obtain the following

$$\|U(t)\| \le \frac{M_{1}}{t}\int_{0}^{\infty} e_{1,\alpha}^{1,0}(-st^{-\alpha})\, e^{\omega s}ds\int_{0}^{\infty}\exp(-\omega_{0}s\tau^{r})\,\frac{d\tau}{\tau^{r}}. \tag{8}$$

In the inner integral of (8) we make the substitution $\omega_{0}s\tau^{r} = \xi$. Then,

$$\|U(t)\| \le \frac{M_2}{t} \int_0^\infty s^{1-1/r} \; e_{1,\alpha}^{1,0}(-st^{-\alpha}) \; e^{\omega s} ds \int_0^\infty e^{-\xi} \xi^{1/r-2} d\xi =$$

$$= \frac{M_3}{t} \int_0^\infty s^{1-1/r} \; e_{1,\alpha}^{1,0}(-st^{-\alpha}) \; e^{\omega s} ds.$$

In the last integral of (9), we divide the interval of integration by the number $b > 0$ and estimate each of the integrals obtained. We will have the following:

$$\|U(t)\| \le \frac{M_3 \, e^{\omega b}}{t} \int_0^b s^{1-1/r} \; e_{1,\alpha}^{1,0}(-st^{-\alpha}) \, ds + \frac{M_3 \, b^{1-1/r}}{t} \int_b^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) e^{\omega s} ds \le$$

$$\le \frac{M_3 \, e^{\omega b}}{t} \int_0^\infty s^{1-1/r} \; e_{1,\alpha}^{1,0}(-st^{-\alpha}) \, ds + \frac{M_3 \, b^{1-1/r}}{t} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) e^{\omega s} ds. \qquad (10)$$

The integrals in (10) are calculated (see (10), the equation (2.2.3) and (2.2.31) and, taking into account the asymptotics of the Mittag-Leffler function at $0 < \beta < 2, \nu \in (\beta/2, \beta)$ and $|z| \to \infty$

$$E_{\beta,\mu}(z) = \frac{1}{\beta} z^{(1-\mu)/\beta} \exp\left(z^{1/\beta}\right) - \sum_{j=1}^n \frac{z^{-j}}{\Gamma(\mu - \beta j)} + O\left(\frac{1}{|z|^{n+1}}\right), \qquad |\arg z| \le \nu\pi,$$

$$E_{\beta,\mu}(z) = -\sum_{j=1}^n \frac{z^{-j}}{\Gamma(\mu - \beta j)z^j} + O\left(\frac{1}{|z|^{n+1}}\right), \qquad \nu\pi \le |\arg z| \le \pi,$$

we obtain the inequality

$$\|U(t)\| \le M_4 \left(t^{\alpha(2-1/r)-1} + t^{\alpha-1} E_{\alpha,\alpha}(\omega t^\alpha)\right) \le M_5 \, t^{\alpha(2-1/r)-1} \exp(\omega^{1/\alpha} t). \qquad (11)$$

Hence, $U(t)$ is a continuous operator-valued function in $L(E)$ at $t > 0$. Moreover, at $t > 0$ it has a continuous fractional derivative. Indeed, since $|\lambda| \le M_6(1 + \tau)$, then, similarly to the estimates (7) – (11) we obtain the following:

$$\left\|D^\alpha U(t)\right\| \le \frac{M_1}{t} \int_0^\infty \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) \exp(\omega s - \omega_0 s(1+\tau)^r) \, ds \, \frac{d\tau}{(1+\tau)^{r-1}} \le$$

$$\leq \frac{M_1}{t} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) e^{\omega s} ds \int_0^\infty \exp(-\omega_0 s(1+\tau)^r) \; \frac{d\tau}{(1+\tau)^{r-1}} \leq$$

$$\leq \frac{M_3}{t} \int_0^\infty s^{1-2/r} e_{1,\alpha}^{1,0}(-st^{-\alpha}) e^{\omega s} ds \leq$$

$$\leq \frac{M_3 e^{\omega b}}{t} \int_0^\infty s^{1-2/r} e_{1,\alpha}^{1,0}(-st^{-\alpha}) ds + \frac{M_3 b^{1-2/r}}{t} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) e^{\omega s} ds \leq$$

$$\leq M_4 \left(t^{\alpha(2-2/r)-1} + t^{\alpha-1} E_{\alpha,\alpha}(\omega t^\alpha) \right).$$

Thus,

$$\left\| D^\alpha U(t) \right\| \leq M_5 \, t^{\alpha(2-2/r)-1} \exp(\omega^{1/\alpha} t). \tag{12}$$

Let's note that integrals were used to obtain the estimate (12) (see (10), the equation (2.2.3)

$$\int_0^\infty s^{-1} e_{1,\alpha}^{1,0}(-st^{-\alpha}) ds = \alpha, \qquad \int_0^\infty s^{1-2/r} e_{1,\alpha}^{1,0}(-st^{-\alpha}) ds = \frac{\Gamma(2-2/r)}{\Gamma(2\alpha - 2\alpha/r)} t^{2\alpha - 2\alpha/r}.$$

The last integral converges at $2 - 2/r > -1$, therefore the term $2/3 < r \leq 1$ is imposed in the inequal-

ity. Let's note that if $\alpha = 1$, then this condition does not arise for the parameter r (see Sil'chenko,

2002) and for a first-order equation it can be $0 < r < 1$.

Theorem 1. *Suppose that the term 1 is satisfied. Then at any* $u_0 \in E$ *the function* $u(t) = B^{-1} U(t) u_0$

is the solution of the problem (2), (3) and the following estimate is fair at that

$$\left\| u(t) \right\| \leq M_7 \, t^{\alpha(2-1/r)-1} \exp(\omega^{1/\alpha} t), \quad M_7 > 0. \tag{13}$$

Proof. The operator D^α can be applied to the operator function $U(t)$ defined by the equality (4).

After obvious transformations, using the equality (5), we obtain the following:

$$D^\alpha U(t) = \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \lambda E_{\alpha,\alpha}(t^\alpha \lambda) A(\lambda A - B)^{-1} d\lambda =$$

$$= \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) \; d\lambda + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$\frac{1}{2\pi i t} \int_\gamma \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) e^{\lambda s} \, ds \, d\lambda + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$\frac{1}{2\pi i t} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) \, ds \int_\gamma e^{\lambda s} \; d\lambda + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$= \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda .$$

Therefore, $AB^{-1}D^\alpha U(t) = U(t)$, thus $AD^\alpha u(t) = Bu(t)$ and the function $u(t)$ satisfies the equation

(2).

Let's show that the function $u(t)$ satisfies the initial condition (3). Using (5), we put down $U(t)$ in

the following form:

$$U(t) = \frac{t^{\alpha-1}}{2\pi i} \int_\gamma E_{\alpha,\alpha}(t^\alpha \lambda) A(\lambda A - B)^{-1} \; d\lambda =$$

$$= \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \frac{1}{\lambda} E_{\alpha,\alpha}(t^\alpha \lambda) \; d\lambda + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \frac{1}{\lambda} E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$= \frac{1}{2\pi i t} \int_\gamma \frac{1}{\lambda} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) \, e^{\lambda s} \, ds \, d\lambda + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \frac{1}{\lambda} E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$= \frac{1}{2\pi i t} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) \, ds \int_\gamma \frac{1}{\lambda} e^{\lambda s} \; d\lambda + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \frac{1}{\lambda} E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$= \frac{1}{t} \int_0^\infty e_{1,\alpha}^{1,0}(-st^{-\alpha}) \, ds + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \frac{1}{\lambda} E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda =$$

$$= \frac{t^{\alpha-1}}{\Gamma(\alpha)} I + \frac{t^{\alpha-1}}{2\pi i} \int_\gamma \frac{1}{\lambda} E_{\alpha,\alpha}(t^\alpha \lambda) B(\lambda A - B)^{-1} \; d\lambda .$$

To this equality we apply the bounded operator $AB^{-1}I^{1-\alpha}$. We will have the following:

$$AB^{-1}I^{1-\alpha}U(t) = AB^{-1} + \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda} E_{\alpha,1}(t^\alpha \lambda) A(\lambda A - B)^{-1} \, d\lambda. \qquad (15)$$

The integral in (15) converges absolutely and at $t \to 0$ it tends to the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda} A(\lambda A - B)^{-1} \, d\lambda, \qquad (16)$$

which, as we shall show below, is equal to zero. Indeed, let's consider the circle $\sigma^2 + \tau^2 = R^2$ and denote its part via Γ_R, that lies to the right of the curve γ, and via γ_R – the part of the curve γ, lying inside the circle of the radius R. The integral of an analytic function $\lambda^{-1}A(\lambda A - B)^{-1}$ over a closed contour $\gamma_R \cup \Gamma_R$ makes zero. We will stretch this contour, striving $R \to \infty$. At that, in view of the estimate (1)

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_R} \frac{1}{\lambda} A(\lambda A - B)^{-1} \, d\lambda \right\| \le \frac{1}{2\pi i}\int_{-\varphi}^{\varphi} \frac{1}{R}\cdot\frac{M}{R^r}\cdot R \, d\varphi = \frac{M\varphi}{\pi R^r} < \frac{M}{R^r}, \qquad (18)$$

where $\varphi \in (\pi/2, \pi)$ – the angle of inclination of the tangent to the curve γ at $\tau \to +0$.

From the inequality (17) at $R \to \infty$ the equality equal to integral zero follows (16), and from (15) the following equality follows:

$$\lim_{t\to 0} AB^{-1}I^{1-\alpha}U(t) = AB^{-1}, \qquad (18)$$

and, thus,

$$\lim_{t\to 0} AI^{1-\alpha}u(t) = AB^{-1}u_0,$$

i.e., the function $u(t)$ also satisfies the initial term (3).

Finally, the estimate (13) follows from (11). The theorem is proved.

Theorem 2. *Suppose that term 1 is satisfied and the operator A is closed. Then at any $u_0 \in E$ the problem (2), (3) has a unique solution in the class of functions that satisfy the inequality (13).*

Proof. Suppose that the function $v(t)$ satisfies the equation (2), zero initial condition (3) and admits the estimate (13). Then we can consider the function

$$w(t, \lambda) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) v(s)\, ds, \qquad \operatorname{Re}\lambda > \omega.$$

We apply the operator B to the function $w(t, \lambda)$ and taking into account the fractional integration formula by parts see (Samko, et al. 1993), formula (2.64), we obtain the following

$$Bw(t, \lambda) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) Bv(s)\, ds =$$

$$= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) A D^\alpha v(s)\, ds = \int_0^t D_{t-}^\alpha \left((t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\right) A v(s)\, ds =$$

$$= \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) A v(s)\, ds = \lambda A w(t, \lambda),$$

at that we used the notation $D_{b-}^\alpha u(t) = \left(-\dfrac{d}{dt}\right) I_{b-}^{1-\alpha} u(t)$ – the right-hand fractional Riemann-Liouville derivative, and $I_{b-}^{1-\alpha} u(t) = \dfrac{1}{\Gamma(1-\alpha)} \int_t^b \dfrac{u(\tau)}{(\tau-t)^\alpha}\, d\tau$ – the right-hand fractional Riemann-Liouville integral.

Thus, $(\lambda A - B)w(t, \lambda) \equiv 0$, and since λ belongs to A - resolvent set of the operator B, then

$$w(t, \lambda) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) v(s)\, ds \equiv 0, \qquad \operatorname{Re}\lambda > \omega. \tag{19}$$

Let's consider an integral equation with an unknown function $\varphi(t)$

$$\varphi(t) = v(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) \, ds , \qquad (20)$$

which has a unique solution (see Rance, 2009 p. 123), which, taking into account (19), is representable in the following form:

$$\varphi(t) = v(t) + \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) v(s) \, ds = v(t) . \qquad (21)$$

From the equations (20), (21) we derive the equality $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \, ds = I^\alpha v(t) \equiv 0$, from

which, after the application of the operator D^α we obtain the set $v(t) \equiv 0$. The theorem is proved.

Let then $f(t)$ – is a given function with the values in E and $u_0 \in E$. Let's consider the problem of

the function $u(t)$ determination, satisfying the inhomogeneous equation:

$$AD^\alpha u(t) = Bu(t) + AB^{-1} f(t), \qquad (22)$$

And the initial term

$$\lim_{t \to 0} AI^{1-\alpha} u(t) = AB^{-1} u_0 . \qquad (23)$$

Theorem 3. *Let's assume that the term 1 is satisfied, the operator A is closed, and the function*

$f(t) \in C((0,\infty), E)$ *is absolutely integrable at zero and* $u_0 \in E$. *Then there is a unique solution of*

the problem (22), (23), which has the following form

$$u(t) = B^{-1} U(t) u_0 + \int_0^t B^{-1} U(t-s) f(s) \, ds , \qquad (24)$$

where the operator function $U(t)$ *is defined by (4).*

Proof. It is sufficient to verify that the function

$$w(t) = \int_0^t B^{-1} U(t-s) f(s) \, ds$$

satisfies the equation (22) and the nonzero initial condition (23).

Taking into account (14), (18), after obvious transformations we will have

$$AD^{\alpha}w(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} \int_0^{\tau} AB^{-1}U(t-s)f(s)\,ds\,d\tau =$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \int_s^t (t-\tau)^{-\alpha} AB^{-1}U(t-s)f(s)\,d\tau\,ds =$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \int_0^{t-s} (t-s-\xi)^{-\alpha} AB^{-1}U(\xi)f(s)\,d\xi\,ds \ . \tag{25}$$

In (25), under the integral sign s the continuous $t-s$ function is obtained, therefore

$$AD^{\alpha}w(t) = \frac{1}{\Gamma(1-\alpha)} \lim_{s\to t} \int_0^{t-s} (t-s-\xi)^{-\alpha} AB^{-1}U(\xi)f(s)\,d\xi +$$

$$+ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d}{dt} \int_0^{t-s} (t-s-\xi)^{-\alpha} AB^{-1}U(\xi)f(s)\,d\xi\,ds =$$

$$= \lim_{t-s\to 0} AB^{-1}I^{1-\alpha}U(t-s)f(s) + \int_0^t AB^{-1}D^{\alpha}U(t-s)f(s)\,ds =$$

$$= AB^{-1}f(t) + B\int_0^t B^{-1}U(t-s)f(s)\,ds = AB^{-1}f(t) + Bw(t),$$

Thus, the function $w(t)$ meets the equation (22).

We further believe that the function $w(t)$ satisfies the zero-initial condition (23).

Using the properties of fractional integration, we obtain the following

$$\lim_{t\to+0} AI^{1-\alpha}w(t) = \frac{1}{\Gamma(1-\alpha)} \lim_{t\to+0} \int_0^t (t-\tau)^{-\alpha} \int_0^{\tau} AB^{-1}U(\tau-\xi)f(\xi)\,d\xi\,d\tau =$$

$$= \frac{1}{\Gamma(1-\alpha)} \lim_{t\to+0} \int_0^t \int_0^{t-\xi} (t-\xi-s)^{-\alpha} AB^{-1}U(s)f(\xi)\,ds\,d\xi =$$

$$= \lim_{t \to +0} \int_0^t AB^{-1} I^{1-\alpha} U(t-\xi) f(\xi) \, d\xi \; , \tag{26}$$

It follows from the equation (26), (18) that

$$\lim_{t \to +0} AI^{1-\alpha} w(t) = 0.$$

Thus, the function $w(t)$ satisfies the zero-initial condition (23), and the function $u(t)$ defined by (24) is the unique solution of the problem (22), (23). The theorem is proved.

2. Inverse problem of Cauchy type. Then let's consider the problem of the function $v(t)$ determination, belonging to $D(B)$ at $t \in (0, 1]$ and the element $AB^{-1} p \in E$ from the terms

$$AD^\alpha v(t) = B v(t) + t^{k-1} AB^{-1} p \, , \tag{27}$$

$$\lim_{t \to 0} AI^{1-\alpha} v(t) = AB^{-1} v_0 \, , \tag{28}$$

$$\lim_{t \to 1} I^\beta v(t) = v_1 \, , \tag{29}$$

where $k > 0$, $\beta \geq 0$ (I^β is a single operator at $\beta = 0$). The interval $t \in (0, 1]$ is chosen for a more compact recording of the formulas.

Definition 2. The solution of the problem (27) – (29) is the pair $\left(v(t), AB^{-1} p\right)$, for which $v(t) \in C((0, 1], D(B))$, $I^{1-\alpha} v(t) \in C^1((0, 1], D(A))$, $AB^{-1} p \in E$, and at that $v(t)$ and $AB^{-1} p$ satisfy the equations (27) – (29).

The considered problem (27) - (29) is called the inverse problem here, in contrast to the direct problem of Cauchy type (27), (28) with the known element $AB^{-1} p \in E$. It can also be interpreted as the restoration in the equation (27) of a nonstationary term $t^{k-1} AB^{-1} p$ by the means of the additional boundary condition (29). A review of publications on inverse problems for the equations of integer order can be found in (Prilepko, et al. 2000), the inverse problem for the equation with the fraction-

al Riemann-Liouville derivative was studied in (Glushak, 2010), and the degenerate inverse problem (27) - (29) is considered for the first time.

As follows from Theorem 3, in order to solve the inverse problem (27) - (29), it is necessary to find the function $v(t)$ and the parameter $AB^{-1}p$ such that the following equation was true:

$$v(t) = B^{-1}U(t)v_0 + \int_0^t s^{k-1}B^{-1}U(t-s)p \, ds \tag{30}$$

and the equation (29) is carried out.

From (30) and the boundary condition (29) we obtain the equation $\lim_{t\to1} B^{-1}I^{\beta}U(t)v_0 + \Gamma(k)\lim_{t\to1} B^{-1}I^{\beta+k}U(t)p = v_1$ that we rewrite in the following form to find the unknown element $AB^{-1}p$:

$$Q \, p = q, \tag{31}$$

где $q = \dfrac{1}{\Gamma(k)}\left(v_1 - B^{-1}I^{\beta}U(1)v_0\right),$ $\qquad Q_p = \dfrac{1}{\Gamma(k+\beta)}\int_0^1 (1-s)^{k+\beta-1}B^{-1}U(s)p \, ds =$

$$= \dfrac{1}{\Gamma(k+\beta)}\int_0^1 (1-s)^{k+\beta-1}\dfrac{s^{\alpha-1}}{2\pi i}\int_\gamma E_{\alpha,\alpha}(s^\alpha\lambda)B^{-1}A(\lambda A - B)^{-1}p \, d\lambda \, ds =$$

$$= \dfrac{1}{2\pi i}\int_\gamma \lim_{s\to1} I^{\beta+k}(s^{\alpha-1}E_{\alpha,\alpha}(s^\alpha\lambda))B^{-1}A(\lambda A - B)^{-1}p \, d\lambda =$$

$$= \dfrac{1}{2\pi i}\int_\gamma E_{\alpha,\alpha+\beta+k}(\lambda)B^{-1}A(\lambda A - B)^{-1}p \, d\lambda \; . \tag{32}$$

Thus, the unique solvability of the inverse problem (27) - (29) is reduced to the problem of a unique solution existence $AB^{-1}p$ of the equation (31) with the operator Q set by (32).

Just as in inverse problems for nondegenerate equations, in the case under consideration, the location of the zeros of some analytic function plays an important role in establishing the solvability, in

our case it is the function $E_{\alpha,\alpha+\beta+k}(z)$. Therefore, we give the results of (Sedletskiĭ, 2000; Suleri, and Cavagnaro, 2016) we need about their disposition. In Theorem 1 (Sedletskiĭ, 2000; Bernasconi, and Emilio, 2018) it was established that at $\alpha \in (0, 1)$, $\alpha + \beta + k > 0$ and a suitable numbering, all sufficiently large zeros by modulus μ_n, $n \in Z \setminus \{0\}$ of the function $E_{\alpha,\alpha+\beta+k}(z)$ are simple and at $n \to \pm\infty$ the following asymptotics is true:

$$\mu_n^{1/\alpha} = 2\pi n i + (k + \beta - 1)\left(\ln 2\pi |n| + \frac{\pi i}{2} \operatorname{sign} n \right) + \ln \frac{\alpha}{\Gamma(k + \beta)} + o(1), \quad n \to \pm\infty. \tag{33}$$

As we noted earlier, the generalized resolvent $(\lambda A - B)^{-1}$ of the operator B is defined in the region Ξ, bounded by the curve γ to the left with the equation $\sigma = \omega - \omega_0 (1 + \tau^2)^{r/2}$, $\omega > \omega_0 > 0$. Taking into account the asymptotics of zeros μ_n of the function $E_{\alpha,\alpha+\beta+k}(z)$, defined by equality (33), we can assert that outside Ξ there can be only a finite number of zeros μ_n, $n = 1, 2, \ldots, n_0$.

Let's assume that all these zeros $\mu_n, n = 1, 2, \ldots, n_0$ belong to A-resolvent set $\rho_A(B)$. Let's surround each of these points by a circular neighborhood $\gamma_n, n = 1, 2, \ldots, n_0$ of such a small radius that there are no points in A spectrum of the operator B, that is, the points from $\sigma_A(B) = C \setminus \rho_A(B)$. This can be done, since the A-resolvent set of the operator B is open. THUS, A-spectrum of the operator B is located in the region bounded by the lines γ and $\gamma_n, n = 1, 2, \ldots, n_0$. Let's denote by

$\Gamma = \tilde{\gamma} \bigcup\limits_{n=1}^{n_0} \gamma_n$, where $\tilde{\gamma}$ is the contour γ shifted to the right, the circles γ_n are done clockwise.

In order to prove the existence of a unique solution $AB^{-1}p$ for equation (31), we introduce the following operator for consideration:

$$P_x = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{E_{\alpha,\beta+k}(z) \, A(zA - B)^{-1} x}{E_{\alpha,\alpha+\beta+k}(z)} \, dz, \quad x \in E. \tag{34}$$

Let's note that, similarly to (12), it is proved that the integral is absolutely convergent in (34) by the choice of the contour Γ, the inequality (1) and the asymptotic behavior of the Mittag-Leffler function at $0 < \alpha < 2$ и $|z| \to \infty$.

Let $x \in E$, then substituting (34) into (32), and using the identity

$$(\lambda A - B)^{-1} A(zA - B)^{-1} = \frac{(\lambda A - B)^{-1} - (zA - B)^{-1}}{z - \lambda},$$

We obtain the following equation:

$$QPx = \frac{1}{2\pi i} \int_\gamma E_{\alpha,\alpha+\beta+k}(\lambda) \, B^{-1} A(\lambda A - B)^{-1} \frac{1}{2\pi i} \int_\Gamma \frac{E_{\alpha,\beta+k}(z) \, A(zA - B)^{-1} x}{E_{\alpha,\alpha+\beta+k}(z)} \, dz \, d\lambda =$$

$$+ \frac{1}{2\pi i} \int_\gamma E_{\alpha,\alpha+\beta+k}(\lambda) B^{-1} A(\lambda A - B)^{-1} x \, \frac{1}{2\pi i} \int_\Gamma \frac{E_{\alpha,\beta+k}(z) \, dz}{E_{\alpha,\alpha+\beta+k}(z)(z - \lambda)} +$$

$$+ \frac{1}{2\pi i} \int_\gamma E_{\alpha,\alpha+\beta+k}(\lambda) \, \frac{1}{2\pi i} \int_\Gamma \frac{E_{\alpha,\beta+k}(z) B^{-1} A(zA - B)^{-1} x \, dz}{E_{\alpha,\alpha+\beta+k}(z)(\lambda - z)} =$$

$$= \frac{1}{2\pi i} \int_\gamma E_{\alpha,\alpha+\beta+k}(\lambda) \, \frac{1}{2\pi i} \int_\Gamma \frac{E_{\alpha,\beta+k}(z) B^{-1} A(zA - B)^{-1} x \, dz}{E_{\alpha,\alpha+\beta+k}(z)(\lambda - z)} . \tag{35}$$

The integral is absolutely convergent in (35). Changing the order of integration, we will have the following:

$$QPx = \frac{1}{2\pi i} \int_\Gamma \frac{E_{\alpha,\beta+k}(z) B^{-1} A(zA - B)^{-1} x}{E_{\alpha,\alpha+\beta+k}(z)} \, \frac{1}{2\pi i} \int_\gamma \frac{E_{\alpha,\alpha+\beta+k}(\lambda)}{(\lambda - z)} \, d\lambda \, dz =$$

$$= \frac{1}{2\pi i} \int_\Gamma E_{\alpha,\beta+k}(z) B^{-1} A(zA - B)^{-1} x \, dz = \frac{1}{2\pi i} \int_{\tilde\gamma} E_{\alpha,\beta+k}(z) B^{-1} A(zA - B)^{-1} x \, dz =$$

$$= B^{-1} I^{k+\beta-\alpha} \, U(1)x .$$

The expression similar to (36) appears if we consider the element BQx. Indeed, taking into account the equalities (32), (14), (18) and assuming $k + \beta - 1 > 0$, we obtain the following:

$$BQx = \frac{1}{\Gamma(k+\beta)} \int_0^1 (1-s)^{k+\beta-1} U(s) x \, ds = \frac{AB^{-1}}{\Gamma(k+\beta)} \int_0^1 (1-s)^{k+\beta-1} \frac{d}{ds} I^{1-\alpha} U(s) x \, ds =$$

$$= \frac{AB^{-1}}{\Gamma(k+\beta)} \left((1-s)^{k+\beta-1} I^{1-\alpha} U(s) \left. x \right|_0^1 + (k+\beta-1) \int_0^1 (1-s)^{k+\beta-2} I^{1-\alpha} U(s) \, x \, ds \right) =$$

$$= -\frac{1}{\Gamma(k+\beta)} AB^{-1} x + AB^{-1} I^{k+\beta-\alpha} U(1) \, x. \tag{37}$$

Further analysis of the equation (31) solvability will be carried out when the following condition is satisfied.

Term 2. Let $k + \beta - 1 > 0$ and on $D(B)$ the operators $A, \ B^{-1}, (\lambda A - B)^{-1}$ commute.

If the term 2 is satisfied, then, introducing the notation $p_1 = AB^{-1} p$,

$$Q_1 p_1 = \frac{1}{2\pi i} \int_\gamma E_{\alpha,\alpha+\beta+k}(\lambda)(\lambda A - B)^{-1} p_1 \, d\lambda, \tag{38}$$

put down the equation (31) in the following form

$$Q_1 p_1 = q.$$

Let's consider the operator $\tilde{Q} = \Gamma(k+\beta)(AP - B)$. By virtue of equalities (36), (37) and condition 1 for $p_1 \in D(B)$ the following relations hold:

$$Q_1 \tilde{Q} \, p_1 = \Gamma(k+\beta)(Q_1 AP - Q_1 B) p_1 = \Gamma(k+\beta)(AQP - BQ)p = AB^{-1} p = p_1, \tag{40}$$

$$\tilde{Q} Q_1 p_1 = \Gamma(k+\beta)(APQ_1 - BQ_1) p_1 = \Gamma(k+\beta)(AQP - BQ)p = AB^{-1} p = p_1. \tag{41}$$

The equations (40), (41) allow to prove the following criterion.

Theorem 4. *Let the operator A is closed and the conditions 1 and 2 are fulfilled. In order that the inverse problem (27) - (29) at any $v_0 \in E, \quad v_1 \in D(B)$ is uniquely solvable, it is necessary and sufficient that every point $\mu_n, n = 1, 2, \ldots, n_0$, that is the zero of the function $E_{\alpha,\alpha+\beta+k}(z)$ and lying to the left of γ, belongs ti A - resolvent set $\rho_A(B)$.*

Proof. Let us prove a sufficient condition of the theorem. If the solution of the equation (39) exists, then, as follows from (41), it is unique and has the following form:

$$p_1 = AB^{-1}p = \tilde{Q}q = \frac{1}{\Gamma(k)}\left(v_1 - B^{-1}I^{\beta}U(1)v_0\right),$$

and it follows from (40) that $p_1 = AB^{-1}p$ is the solution of the equation (39).

The function $v(t)$ is defined by the equation (30) in the following way:

$$v(t) = B^{-1}U(t)v_0 + \int_0^t s^{k-1}B^{-1}U(t-s)p \, ds =$$

$$= B^{-1}U(t)v_0 + \frac{1}{2\pi i}\int_0^t s^{k-1}(t-s)^{\alpha-1}\int_{\gamma} E_{\alpha,\alpha}((t-s)^{\alpha}\lambda)(\lambda A - B)^{-1}AB^{-1}p \, d\lambda \, ds.$$

Proof of necessity. Let $\mu_n, n = 1, 2, \ldots, n_0$ is an arbitrary point that is the zero of the function $E_{\alpha,\alpha+\beta+k}(z)$ and located to the left of γ. Let's consider the bounded operator S_n, given by the equation

$$S_n = \frac{1}{2\pi i}\int_{\gamma} \frac{E_{\alpha,\alpha+\beta+k}(z)}{\mu_n - z}(zA - B)^{-1} \, dz.$$

Integrating along the contour γ both sides of the identity

$$\frac{E_{\alpha,\alpha+\beta+k}(z)}{\mu_n - z}(\mu_n A - B)(zA - B)^{-1} = E_{\alpha,\alpha+\beta+k}(z)A(zA - B)^{-1} + \frac{E_{\alpha,\alpha+\beta+k}(z)}{\mu_n - z},$$

Due to the closedness of the operators A and B and the analyticity of the function $\dfrac{E_{\alpha,\alpha+\beta+k}(z)}{\mu_n - z}$ in the domain located from γ and to the left of (38), we arrive to the following relation:

$$(\mu_n A - B)S_n = Q_1. \tag{42}$$

If the inverse problem (27) - (29) at any $v_0 \in E$, $v_1 \in D(B)$ is uniquely solvable, then the operator Q_1 has an inverse operator $\tilde{Q}$ determined along the whole $D(B)$. Then, due to (42), the following equalities are true:

$$x = \tilde{Q}Q_1 x = \tilde{Q}(\mu_n A - B)S_n x = (\mu_n A - B)\tilde{Q}S_n x, \quad x \in E,$$

$$x = Q_1\tilde{Q}x = (\mu_n A - B)S_n\tilde{Q}x = \tilde{Q}S_n(\mu_n A - B)x, \quad x \in D(B).$$

Thus, the operator $\tilde{Q}S_n$ is defined on the whole E and is inverse operator with respect to $(\mu_n A - B)$. Therefore, μ_n belongs to A-resolvent set $\rho_A(B)$ and the necessity, and, thus, the theorem is proved.

CONCLUSIONS.

According to the operator coefficients of the equation, they designed some operator function - the resolving operator belonging to the space of bounded linear operators, and with its help they established a unique solvability of Cauchy-type problem for a degenerate fractional abstract equation with a non-densely defined operator coefficient.

Using the designed resolving operator, it was possible to study also the unique solvability of a corresponding inverse problem.

An example of operators is given satisfying the conditions imposed in the work.

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